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Proper holomorphic mappings between generalized Hartogs triangles

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Abstract Answering all questions—concerning proper holomorphic mappings between generalized Hartogs triangles—posed by Jarnicki and Plfug (First steps in several complex variables: Reinhardt domains, EMS Textbooks in Mathematics, European Mathematical Society Publishing House, 2008), we characterize the existence of proper holomorphic mappings between generalized Hartogs triangles and give their explicit form. In particular, we completely describe the group of holomorphic automorphisms of such domains and establish rigidity of proper holomorphic self-mappings on them. Moreover, we also complete the classification of proper holomorphic mappings in the class of complex ellipsoids which was initiated by Landucci and continued by Dini and Selvaggi Primicerio.

Keywords Generalized Hartogs triangle · Proper holomorphic mapping · Group of automorphisms · Complex ellipsoid

Mathematics Subject Classification 32H35

1 Introduction

In the paper, we study proper holomorphic mappings between generalized Hartogs triangles of equal dimensions (see the definition below) giving a full characterization of the existence of such mappings, their explicit forms, and a complete description of the group of holomorphic automorphisms of such domains. Our results answer all questions posed by Jarnicki and

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Pflug in [9], Sections 2.5.2 and 2.5.3, concerning proper holomorphic mappings between generalized Hartogs triangles and holomorphic automorphisms of such domains.

Let us recall the definition of the above mentioned domains. Let $n, m \in \mathbb{N}$. For $p = (p_1, \dots, p_n) \in \mathbb{R}^n_{>0}$ and $q = (q_1, \dots, q_m) \in \mathbb{R}^m_{>0}$ define the *generalized Hartogs triangle* as

$$\mathbb{F}_{p,q} := \left\{ (z, w) \in \mathbb{C}^n \times \mathbb{C}^m : \sum_{j=1}^{n} |z_j|^{2p_j} < \sum_{j=1}^{m} |w_j|^{2q_j} < 1 \right\}.$$

Note that $\mathbb{F}_{p,q}$ is a non-smooth pseudoconvex Reinhardt domain, with the origin on the boundary. Moreover, if $n = m = 1$, then $\mathbb{F}_{1,1}$ is the standard Hartogs triangle.

Let $p \in \mathbb{R}^n_{>0}$, $q \in \mathbb{R}^m_{>0}$ and $\tilde{p} \in \mathbb{R}^{\tilde{n}}_{>0}$, $\tilde{q} \in \mathbb{R}^{\tilde{m}}_{>0}$. We say that two generalized Hartogs triangles $\mathbb{F}_{p,q}$ and $\mathbb{F}_{\tilde{p},\tilde{q}}$ are *equidimensional*, if $n = \tilde{n}$ and $m = \tilde{m}$.

The problem of characterization of proper holomorphic mappings

$$\mathbb{F}_{p,q} \longrightarrow \mathbb{F}_{\tilde{p},\tilde{q}} \tag{1}$$

and the group $\mathrm{Aut}(\mathbb{F}_{p,q})$ of holomorphic automorphisms of $\mathbb{F}_{p,q}$ has been investigated in many papers (see, e.g., [12], [5], [6], [2], [3] for the equidimensional case and [4] for the non-equidimensional one). It was Landucci who considered the mappings (1) in 1989 as examples of proper holomorphic mappings between non-smooth pseudoconvex Reinhardt domains, with the origin on the boundary, which do not satisfy a regularity property for the Bergman projection (the so-called R-condition). In [12], he gave a complete characterization of the existence and found explicit forms of mappings (1) in the case $m = 1$, $p, \tilde{p} \in \mathbb{N}^n$, and $q, \tilde{q} \in \mathbb{N}$. Then, in 2001, Chen and Xu (cf. [5]) characterized the existence of mappings (1) for $n > 1$, $m > 1$, $p, \tilde{p} \in \mathbb{N}^n$, and $q, \tilde{q} \in \mathbb{N}^m$. The next step was made one year later, when the same authors fully described proper holomorphic self-mappings of $\mathbb{F}_{p,q}$ for $n > 1$, $m > 1$, $p \in \mathbb{N}^n$, and $q \in \mathbb{N}^m$ (cf. [6]). In the same year, Chen in [2] characterized the existence of mappings (1) in the case $n > 1$, $m > 1$, $p, \tilde{p} \in \mathbb{R}^n_{>0}$, and $q, \tilde{q} \in \mathbb{R}^m_{>0}$. Finally, Chen and Liu in 2003 gave explicit forms of proper holomorphic mappings $\mathbb{F}_{p,q} \longrightarrow \mathbb{F}_{\tilde{p},\tilde{q}}$ but only for $n > 1$, $m > 1$, $p, \tilde{p} \in \mathbb{N}^n$, and $q, \tilde{q} \in \mathbb{N}^m$ (cf. [3]).

We emphasize that Landucci considered only the case $m = 1$ with exponents being positive integers, whereas Chen, Xu, and Liu obtained some partial results with positive integer or arbitrary real positive exponents under general assumption $n \geq 2$ and $m \geq 2$. Consequently, their results are far from being conclusive for the general setting.

The main aim of this note is to give a complete characterization of the existence of mappings (1), where $n, m \in \mathbb{N}$, $p, \tilde{p} \in \mathbb{R}^n_{>0}$, $q, \tilde{q} \in \mathbb{R}^m_{>0}$, their explicit form, and the description of the group $\mathrm{Aut}(\mathbb{F}_{p,q})$ (cf. Theorems 1, 3, and 5) for arbitrary dimensions and arbitrary positive real exponents. In particular, we obtain a classification theorem on rigidity of proper holomorphic self-mappings of generalized Hartogs triangles (cf. Corollary 7), which generalizes Chen's and Xu's main result from [6].

It is worth pointing out that in the general case neither Landucci's method from [12] (where the assumption $p, \tilde{p} \in \mathbb{N}^n$, $q, \tilde{q} \in \mathbb{N}$ is essential) nor Chen's approach from [2] (where the proof strongly depends on the assumption $m \geq 2$) can be used.

The paper is organized as follows. We start with stating the main results. For the convenience of the reader, we split them into three theorems with respect to dimensions of relevant parts of $\mathbb{F}_{p,q}$. Next we shall discuss proper holomorphic mappings between complex ellipsoids $\mathbb{E}_p$ (cf. Sect. 3, Theorem 9) which will turn out to be quite useful in the sequel and may be interesting in its own right. It should be mentioned that Theorem 9 completes the classification of proper holomorphic mappings between complex ellipsoids which was initiated by

Landucci in 1984 (cf. [11]) and continued by Dini and Selvaggi Primicerio in [7]. The boundary behavior of mappings (1) will also be studied (cf. Sect. 4). In the last section, making use of the description of proper holomorphic mappings between complex ellipsoids (Theorem 9) and the boundary behavior of proper holomorphic mappings between generalized Hartogs triangles (Lemma 11), we shall prove our main results.

Here is some notation. Throughout the paper $\mathbb{D}$ denotes the unit disk in the complex plane, additionally by $\mathbb{T}$ we shall denote the unit circle, ∂D stands for the boundary of the bounded domain $D \subset \mathbb{C}^n$. Let Σ_n denote the group of the permutations of the set $\{1, \dots, n\}$. For $\sigma \in \Sigma_n$, $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ denote $z_\sigma := (z_{\sigma(1)}, \dots, z_{\sigma(n)})$ and $\Sigma_n(z) := \{\sigma \in \Sigma_n : z_\sigma = z\}$. We shall also write $\sigma(z) := z_\sigma$. For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n_{>0}$ and $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{R}^n_{>0}$, we shall write $\alpha\beta := (\alpha_1\beta_1, \dots, \alpha_n\beta_n)$ and $1/\beta := (1/\beta_1, \dots, 1/\beta_n)$. If, moreover, $\alpha \in \mathbb{N}^n$, then

$$\Psi_\alpha(z) := z^\alpha := (z_1^{\alpha_1}, \dots, z_n^{\alpha_n}), \quad z = (z_1, \dots, z_n) \in \mathbb{C}^n.$$

For $\lambda \in \mathbb{C}$, $A \subset \mathbb{C}^n$ let $\lambda A := \{\lambda a : a \in A\}$ and $A_* := A \setminus \{0\}$. Finally, let $\mathbb{U}(n)$ denote the set of unitary mappings $\mathbb{C}^n \longrightarrow \mathbb{C}^n$.

2 Main results

We start with the generalized Hartogs triangles of the lowest dimension.

Theorem 1 *Let $n = m = 1$, $p, q, \tilde{p}, \tilde{q} \in \mathbb{R}_{>0}$.*

(a) There exists a proper holomorphic mapping $\mathbb{F}_{p,q} \longrightarrow \mathbb{F}_{\tilde{p},\tilde{q}}$ if and only if there exist $k, l \in \mathbb{N}$ such that

$$\frac{l\tilde{q}}{\tilde{p}} - \frac{kq}{p} \in \mathbb{Z}.$$

(b) A mapping $F : \mathbb{F}_{p,q} \longrightarrow \mathbb{F}_{\tilde{p},\tilde{q}}$ is proper and holomorphic if and only if

$$F(z, w) = \begin{cases} \left(\zeta z^k w^{l\tilde{q}/\tilde{p} - kq/p}, \xi w^l\right), & \text{if } q/p \notin \mathbb{Q} \\ \left(\zeta z^{k'} w^{l\tilde{q}/\tilde{p} - k'q/p} B\left(z^{p'} w^{-q'}\right), \xi w^l\right), & \text{if } q/p \in \mathbb{Q}, \end{cases} \quad (z, w) \in \mathbb{F}_{p,q},$$

where $\zeta, \xi \in \mathbb{T}$, $k, l \in \mathbb{N}$, $k' \in \mathbb{N} \cup \{0\}$ are such that $l\tilde{q}/\tilde{p} - kq/p \in \mathbb{Z}$, $l\tilde{q}/\tilde{p} - k'q/p \in \mathbb{Z}$, $p', q' \in \mathbb{N}$ are relatively prime with $p/q = p'/q'$, and B is a finite Blaschke product non-vanishing at 0 (if $B \equiv 1$, then $k' > 0$). In particular, there are non-trivial proper holomorphic self-mappings in $\mathbb{F}_{p,q}$.

(c) $F \in \mathrm{Aut}(\mathbb{F}_{p,q})$ if and only if

$$F(z, w) = \begin{cases} (\zeta z, \xi w), & \text{if } q/p \notin \mathbb{N} \\ \left(w^{q/p} \phi\left(z w^{-q/p}\right), \xi w\right), & \text{if } q/p \in \mathbb{N}, \end{cases} \quad (z, w) \in \mathbb{F}_{p,q},$$

where $\zeta, \xi \in \mathbb{T}$ and $\phi \in \mathrm{Aut}(\mathbb{D})$.

Remark 2 (a) A counterpart of Theorem 1 for $p, q, \tilde{p}, \tilde{q} \in \mathbb{N}$ was proved (with minor mistakes) in [12], where it was claimed that a mapping $F : \mathbb{F}_{p,q} \longrightarrow \mathbb{F}_{\tilde{p},\tilde{q}}$ is proper and holomorphic if and only if

$$F(z, w) = \begin{cases} \left(\zeta z^k w^{l\tilde{q}/\tilde{p} - kq/p}, \xi w^l\right), & \text{if } q/p \notin \mathbb{N},\ l\tilde{q}/\tilde{p} - kq/p \in \mathbb{Z} \\ \left(\zeta w^{l\tilde{q}/\tilde{p}} B\left(z w^{-q/p}\right), \xi w^l\right), & \text{if } q/p \in \mathbb{N},\ l\tilde{q}/\tilde{p} \in \mathbb{N} \end{cases}, \quad (2)$$

where $\zeta, \xi \in \mathbb{T}$, $k, l \in \mathbb{N}$, and B is a finite Blaschke product. Nevertheless, the mapping

$$\mathbb{F}_{2,3} \ni (z, w) \longmapsto \left(z^3 w^3 B\left(z^2 w^{-3}\right), w^3\right) \in \mathbb{F}_{2,5},$$

where B is non-constant finite Blaschke product non-vanishing at 0, is proper holomorphic but not of the form (2). In fact, it follows immediately from Theorem 1(b) that for any choice of $p, q, \tilde{p}, \tilde{q} \in \mathbb{N}$ one may find a proper holomorphic mapping $F : \mathbb{F}_{p,q} \longrightarrow \mathbb{F}_{\tilde{p},\tilde{q}}$ having, as a factor of the first component, non-constant Blaschke product non-vanishing at 0.

(b) Theorems 1(a), (b) give a positive answer (modulo Landucci's mistake) to the question posed by Jarnicki and Pflug (cf. [9], Remark 2.5.22 (a)).

(c) Theorem 1(c) gives a positive answer to the question posed by Jarnicki and Pflug (cf. [9], Remark 2.5.15 (b)) in the case $n = 1$.

Our next result is the following

Theorem 3 *Let $n \geq 2$, $m = 1$, $p = (p_1, \ldots, p_n)$, $\tilde{p} = (\tilde{p}_1, \ldots, \tilde{p}_n) \in \mathbb{R}^n_{>0}$, $q, \tilde{q} \in \mathbb{R}_{>0}$.*

(a) There exists a proper holomorphic mapping $\mathbb{F}_{p,q} \longrightarrow \mathbb{F}_{\tilde{p},\tilde{q}}$ if and only if there exist $\sigma \in \Sigma_n$ and $r \in \mathbb{N}$ such that

$$\frac{p_\sigma}{\tilde{p}} \in \mathbb{N}^n \quad \textit{and} \quad \frac{r\tilde{q} - q}{\tilde{p}_j} \in \mathbb{Z}, \quad j = 1, \ldots, n.$$

(b) A mapping $F = (G_1, \ldots, G_n, H) : \mathbb{F}_{p,q} \longrightarrow \mathbb{F}_{\tilde{p},\tilde{q}}$ is proper and holomorphic if and only if

$$\begin{cases} G_j(z, w) = w^{r\tilde{q}/\tilde{p}_j} g_j\left(z_1 w^{-q/p_1}, \ldots, z_n w^{-q/p_n}\right), \quad j = 1, \ldots, n, \\ H(z, w) = \xi w^r, \end{cases} \qquad (z, w) \in \mathbb{F}_{p,q},$$

where $g := (g_1, \ldots, g_n) : \mathbb{E}_p \longrightarrow \mathbb{E}_{\tilde{p}}$ is proper and holomorphic (cf. Theorem 9), $\xi \in \mathbb{T}$, and $r \in \mathbb{N}$ is such that $(r\tilde{q} - q)/\tilde{p}_j \in \mathbb{Z}$, $j = 1, \ldots, n$. Moreover, if there is a j such that $p_{\sigma(j)} \in \mathbb{N}$, $1/\tilde{p}_j \in \mathbb{N}$ and $\{q, \tilde{q}\} \not\subset \mathbb{N}$, then $g(0) = 0$. In particular, there are non-trivial proper holomorphic self-mappings in $\mathbb{F}_{p,q}$.

(c) $F = (G_1, \ldots, G_n, H) \in \mathrm{Aut}(\mathbb{F}_{p,q})$ if and only if

$$\begin{cases} G_j(z, w) = w^{q/p_j} g_j\left(z_1 w^{-q/p_1}, \ldots, z_n w^{-q/p_n}\right), \quad j = 1, \ldots, n, \\ H(z, w) = \xi w, \end{cases} \qquad (z, w) \in \mathbb{F}_{p,q},$$

where $g := (g_1, \ldots, g_n) \in \mathrm{Aut}(\mathbb{E}_p)$ (cf. Theorem 9), $\xi \in \mathbb{T}$. Moreover, if $q \notin \mathbb{N}$, then $g(0) = 0$.

Remark 4 (a) Theorem 3 (a) gives a positive answer to the question posed by Jarnicki and Pflug (cf. [9], Remark 2.5.22 (a)) in the case $n \geq 2$.

(b) Theorem 3 (c) gives a positive answer to the question posed by Jarnicki and Pflug (cf. [9], Remark 2.5.15 (b)) in the case $n \geq 2$.

(c) It should be mentioned that although the structure of the automorphism group $\mathrm{Aut}(\mathbb{F}_{p,q})$ does not change when passing from $p \in \mathbb{N}^n$, $q \in \mathbb{N}$ to $p \in \mathbb{R}^n_{>0}$, $q > 0$, the class of proper holomorphic mappings $\mathbb{F}_{p,q} \longrightarrow \mathbb{F}_{\tilde{p},\tilde{q}}$ does. It is a consequence of the fact that the structure of proper holomorphic mappings $\mathbb{E}_p \longrightarrow \mathbb{E}_{\tilde{p}}$ changes when passing from $p, \tilde{p} \in \mathbb{N}^n$ to $p, \tilde{p} \in \mathbb{R}^n_{>0}$ (see Sect. 3).

Theorem 5 *Let $n, m \in \mathbb{N}$, $m \geq 2$, $p, \tilde{p} \in \mathbb{R}^n_{>0}$, $q, \tilde{q} \in \mathbb{R}^m_{>0}$.*

(a) There exists a proper holomorphic mapping $\mathbb{F}_{p,q} \longrightarrow \mathbb{F}_{\tilde{p},\tilde{q}}$ if and only if there exist $\sigma \in \Sigma_n$ and $\tau \in \Sigma_m$ such that

$$\frac{p_\sigma}{\tilde{p}} \in \mathbb{N}^n \quad \textit{and} \quad \frac{q_\tau}{\tilde{q}} \in \mathbb{N}^m.$$

(b) A mapping $F : \mathbb{F}_{p,q} \longrightarrow \mathbb{F}_{\tilde{p},\tilde{q}}$ is proper and holomorphic if and only if

$$F(z, w) = (g(z), h(w)), \quad (z, w) \in \mathbb{F}_{p,q},$$

where mappings $g : \mathbb{E}_p \longrightarrow \mathbb{E}_{\tilde{p}}$ and $h : \mathbb{E}_q \longrightarrow \mathbb{E}_{\tilde{q}}$ are proper and holomorphic such that $g^{-1}(0) = 0$, $h^{-1}(0) = 0$ (cf. Theorem 9). In particular, if $n = 1$, then there are non-trivial proper holomorphic self-mappings in $\mathbb{F}_{p,q}$; for $n \geq 2$ every proper holomorphic self-mapping in $\mathbb{F}_{p,q}$ is an automorphism.

(c) $F \in \mathrm{Aut}(\mathbb{F}_{p,q})$ if and only if

$$F(z, w) = (g(z), h(w)), \quad (z, w) \in \mathbb{F}_{p,q},$$

where $g \in \mathrm{Aut}(\mathbb{E}_p)$, $h \in \mathrm{Aut}(\mathbb{E}_q)$ with $g(0) = 0$, $h(0) = 0$ (cf. Theorem 9).

Remark 6 (a) Theorem 5 (a) was proved by Chen and Xu in [5] (for $n, m \geq 2$, $p, \tilde{p} \in \mathbb{N}^n$, $q, \tilde{q} \in \mathbb{N}^m$) and by Chen in [2] (for $n, m \geq 2$, $p, \tilde{p} \in \mathbb{R}^n_{>0}$, $q, \tilde{q} \in \mathbb{R}^m_{>0}$).
(b) Theorems 5 (b), (c) were proved by Chen and Xu in [6] for $n, m \geq 2$, $p = \tilde{p} \in \mathbb{N}^n$, $q = \tilde{q} \in \mathbb{N}^m$.
(c) Theorem 5 (c) gives an affirmative answer to the question posed by Jarnicki and Pflug (cf. [9], Remark 2.5.17).

A direct consequence of Theorems 1, 3, and 5 is the following classification of rigid proper holomorphic self-mappings in generalized Hartogs triangles.

Corollary 7 *Let $n, m \in \mathbb{N}$, $p \in \mathbb{R}^n_{>0}$, $q \in \mathbb{R}^m_{>0}$. Then any proper holomorphic self-mapping in $\mathbb{F}_{p,q}$ is an automorphism if and only if $n \geq 2$ and $m \geq 2$.*

Remark 8 Corollary 7 generalizes the main result of [6], where it is proved that for $n \geq 2, m \geq 2$, $p \in \mathbb{N}^n$, and $q \in \mathbb{N}^m$ any proper holomorphic self-mapping in $\mathbb{F}_{p,q}$ is an automorphism. For more information on rigidity of proper holomorphic mappings between special kind of domains in $\mathbb{C}^n$, such as Cartan domains, Hua domains, etc., we refer the reader to [14], [15], [16], [17], and [18].

3 Complex ellipsoids

In this section we discuss proper holomorphic mappings between complex ellipsoids. We shall exploit their form in the proofs of main results.

For $p = (p_1, \ldots, p_n) \in \mathbb{R}^n_{>0}$, define the *complex ellipsoid*

$$\mathbb{E}_p := \left\{ (z_1, \ldots, z_n) \in \mathbb{C}^n : \sum_{j=1}^n |z_j|^{2p_j} < 1 \right\}.$$

Note that $\mathbb{E}_{(1,\ldots,1)}$ is the unit Euclidean ball in $\mathbb{C}^n$. Moreover, if $p/q \in \mathbb{N}^n$, then $\Psi_{p/q} : \mathbb{E}_p \longrightarrow \mathbb{E}_q$ is proper and holomorphic.

The problem of characterization of proper holomorphic mappings between two given complex ellipsoids has been investigated in [11] and [7]. The questions for the existence of such mappings as well as for its form in the case $p, q \in \mathbb{N}^n$ was completely solved by Landucci in 1984 (cf. [11]). The case $p, q \in \mathbb{R}^n_{>0}$ was considered seven years later by Dini and Selvaggi Primicerio in [7], where the authors characterized the existence of proper holomorphic mappings $\mathbb{E}_p \longrightarrow \mathbb{E}_q$ and found $\operatorname{Aut}(\mathbb{E}_p)$. They did not give, however, the explicit form of a proper holomorphic mapping between given two complex ellipsoids. Nevertheless, from the proof of Theorem 1.1 in [7] we easily derive its form which shall be of great importance during the investigation of proper holomorphic mappings between generalized Hartogs triangles.

Theorem 9 *Assume that $n \geq 2$, $p, q \in \mathbb{R}^n_{>0}$.*

(a) (cf. [11], [7]). There exists a proper holomorphic mapping $\mathbb{E}_p \longrightarrow \mathbb{E}_q$ if and only if there exists $\sigma \in \Sigma_n$ such that

$$\frac{p_\sigma}{q} \in \mathbb{N}^n.$$

(b) A mapping $F : \mathbb{E}_p \longrightarrow \mathbb{E}_q$ is proper and holomorphic if and only if

$$F = \Psi_{p_\sigma/(qr)} \circ \phi \circ \Psi_r \circ \sigma,$$

where $\sigma \in \Sigma_n$ is such that $p_\sigma/q \in \mathbb{N}^n$, $r \in \mathbb{N}^n$ is such that $p_\sigma/(qr) \in \mathbb{N}^n$, and $\phi \in \operatorname{Aut}(\mathbb{E}_{p_\sigma/r})$. In particular, every proper holomorphic self-mapping in $\mathbb{E}_p$ is an automorphism.

(c) (cf. [11], [7]). If $0 \leq k \leq n$, $p \in \{1\}^k \times (\mathbb{R}_{>0} \setminus \{1\})^{n-k}$, $z = (z', z_{k+1}, \dots, z_n)$, then $F = (F_1, \dots, F_n) \in \operatorname{Aut}(\mathbb{E}_p)$ if and only if

$$F_j(z) = \begin{cases} H_j(z'), & \text{if } j \leq k \\ \zeta_j z_{\sigma(j)} \left(\frac{\sqrt{1-\|a'\|^2}}{1-\langle z', a'\rangle} \right)^{1/p_{\sigma(j)}}, & \text{if } j > k, \end{cases} \quad z \in \mathbb{E}_p,$$

where $\zeta_j \in \mathbb{T}$, $j > k$, $H = (H_1, \dots, H_k) \in \operatorname{Aut}(\mathbb{B}_k)$, $a' = H^{-1}(0)$, and $\sigma \in \Sigma_n(p)$.

Proof of Theorem 9 Parts (a) and (c) were proved in [7]. (b) Let $F = (F_1, \dots, F_n) : \mathbb{E}_p \longrightarrow \mathbb{E}_q$ be proper and holomorphic. According to [13], any automorphism $H = (H_1, \dots, H_n) \in \operatorname{Aut}(\mathbb{B}_n)$ is of the form

$$H_j(z) = \frac{\sqrt{1-\|a\|^2}}{1-\langle z, a\rangle} \sum_{k=1}^n h_{j,k}(z_k - a_k), \quad z = (z_1, \dots, z_n) \in \mathbb{B}_n, \ j = 1, \dots, n,$$

where $a = (a_1, \dots, a_n) \in \mathbb{B}_n$ and $Q = [h_{j,k}]$ is an $n \times n$ matrix such that

$$\bar{Q}(\mathbb{I}_n - \bar{a}^t a)^t Q = \mathbb{I}_n,$$

where $\mathbb{I}_n$ is the unit $n \times n$ matrix, whereas $\bar{A}$ (resp. ${}^t A$) is the conjugate (resp. transpose) of an arbitrary matrix A. In particular, Q is unitary if $a = 0$.

It follows from [7] that there exists $\sigma \in \Sigma_n$ such that $p_\sigma/q \in \mathbb{N}^n$, $h_{j,\sigma(j)} \neq 0$, and

$$F_j(z) = \left(\frac{\sqrt{1-\|a\|^2}}{1-\langle z^p, a\rangle} h_{j,\sigma(j)} z_{\sigma(j)}^{p_{\sigma(j)}} \right)^{1/q_j} \tag{3}$$

whenever $1/q_j \notin \mathbb{N}$.

If $1/q_j \in \mathbb{N}$, then F_j either is of the form (3), where $p_{\sigma(j)}/q_j \in \mathbb{N}$, or

$$F_j(z) = \left(\frac{\sqrt{1 - \|a\|^2}}{1 - \langle z^p, a \rangle} \sum_{k=1}^{n} h_{j,k}(z_k^{p_k} - a_k) \right)^{1/q_j}$$

where $p_k \in \mathbb{N}$ for any k such that $a_k \neq 0$ or there is a j with $k \neq \sigma(j)$ and $h_{j,k} \neq 0$.

Consequently, if we define $r = (r_1, \ldots, r_n)$ as

$$r_j := \begin{cases} p_{\sigma(j)}, & \text{if } a_{\sigma(j)} \neq 0 \text{ or there is } k \neq \sigma(j) \text{ with } h_{j,k} \neq 0 \\ p_{\sigma(j)}/q_j, & \text{otherwise,} \end{cases}$$

then it is easy to see that $r \in \mathbb{N}^n$, $p_\sigma/(qr) \in \mathbb{N}^n$, and F is as desired. □

Remark 10 (a) The counterpart of Theorem 9 (b) obtained by Landucci in [11] for $p, q \in \mathbb{N}^n$ states that a mapping $F : \mathbb{E}_p \longrightarrow \mathbb{E}_q$ is proper and holomorphic if and only if

$$F = \phi \circ \Psi_{p_\sigma/q} \circ \sigma, \tag{4}$$

where $\sigma \in \Sigma_n$ is such that $p_\sigma/q \in \mathbb{N}^n$ and $\phi \in \mathrm{Aut}(\mathbb{E}_q)$.

(b) In the general case the formula (4) is no longer true (take, for instance, $\Psi_{(2,2)} \circ H \circ \Psi_{(2,2)} : \mathbb{E}_{(2,2)} \longrightarrow \mathbb{E}_{(1/2,1/2)}$, where $H \in \mathrm{Aut}(\mathbb{B}_2)$, $H(0) \neq 0$). In particular, Theorem 9 (b) gives a negative answer to the question posed by Jarnicki and Pflug (cf. [9], Remark 2.5.20).

(c) Note that in the case $p, q \in \mathbb{N}^n$ we have $1/q_j \in \mathbb{N}$ if and only if $q_j = 1$. Hence the above definition of r implies that $r = p_\sigma/q$ and, consequently, Theorem 9 (b) reduces to the Landucci's form (4).

(d) Theorem 9 (c) gives a positive answer to the question posed by Jarnicki and Pflug (cf. [9], Remark 2.5.11).

4 Boundary behavior of proper holomorphic mappings between generalized Hartogs triangles

Note that the boundary $\partial\mathbb{F}_{p,q}$ of the generalized Hartogs triangle $\mathbb{F}_{p,q}$ may be written as $\partial\mathbb{F}_{p,q} = \{0\} \cup K_{p,q} \cup L_{p,q} \cup M_{p,q}$, where

$$K_{p,q} := \left\{ (z, w) \in \mathbb{C}^n \times \mathbb{C}^m : 0 < \sum_{j=1}^{n} |z_j|^{2p_j} = \sum_{j=1}^{m} |w_j|^{2q_j} < 1 \right\},$$

$$L_{p,q} := \left\{ (z, w) \in \mathbb{C}^n \times \mathbb{C}^m : \sum_{j=1}^{n} |z_j|^{2p_j} < \sum_{j=1}^{m} |w_j|^{2q_j} = 1 \right\},$$

$$M_{p,q} := \left\{ (z, w) \in \mathbb{C}^n \times \mathbb{C}^m : \sum_{j=1}^{n} |z_j|^{2p_j} = \sum_{j=1}^{m} |w_j|^{2q_j} = 1 \right\}.$$

Let $\mathbb{F}_{p,q}$ and $\mathbb{F}_{\tilde{p},\tilde{q}}$ be two generalized Hartogs triangles and let $F : \mathbb{F}_{p,q} \longrightarrow \mathbb{F}_{\tilde{p},\tilde{q}}$ be a proper holomorphic mapping. It is known ([12], [5]) that F extends holomorphically through any boundary point $(z_0, w_0) \in \partial\mathbb{F}_{p,q} \setminus \{0\}$.

The aim of this section is to prove the following crucial fact.

Lemma 11 *Let $nm \neq 1$. If $F : \mathbb{F}_{p,q} \longrightarrow \mathbb{F}_{\tilde{p},\tilde{q}}$ is proper and holomorphic, then*

$$F(K_{p,q}) \subset K_{\tilde{p},\tilde{q}} \cup M_{\tilde{p},\tilde{q}}, \quad F(L_{p,q}) \subset L_{\tilde{p},\tilde{q}} \cup M_{\tilde{p},\tilde{q}}.$$

Remark 12 Particular cases of Lemma 11 have already been proved by Landucci (cf. [12], Proposition 3.2, for $p, \tilde{p} \in \mathbb{N}^n$, $q, \tilde{q} \in \mathbb{N}^m$, $m = 1$) and Chen (cf. [2], Lemmas 2.1 and 2.3, for $p, \tilde{p} \in \mathbb{R}^n_{>0}$, $q, \tilde{q} \in \mathbb{R}^m_{>0}$, $m > 1$). Therefore, it suffices to prove Lemma 11 for $n \geq 2$ and $m = 1$. The main difficulty in carrying out this construction is that the methods from [12] (where the assumption $p, \tilde{p} \in \mathbb{N}^n$, $q, \tilde{q} \in \mathbb{N}$ is essential) and [2] (where the assumption $m \geq 2$ is essential) break down. Invariance of two defined parts of boundary of the generalized Hartogs triangles with respect to the proper holomorphic mappings presents a more delicate problem and shall be solved with help of the notion of Levi flatness of the boundary.

The following two lemmas will be needed in the proof of Lemma 11.

Lemma 13 *If $n \geq 2$ and $m = 1$, then $K_{p,q}$ is not Levi flat at $(z, w) \in K_{p,q}$, where at least two coordinates of z are nonzero (i.e., the Levi form of the defining function restricted to the complex tangent space is not degenerate at (z, w)).*

Proof of Lemma 13 Let

$$r(z, w) := \sum_{j=1}^{n} |z_j|^{2p_j} - |w|^{2q}, \quad (z, w) \in \mathbb{C}^n \times \mathbb{C}.$$

Note that r is local defining function for the generalized Hartogs triangle $\mathbb{F}_{p,q}$ (in a neighborhood of any boundary point from $K_{p,q}$). It is easily seen that its Levi form equals

$$\mathcal{L}r((z, w); (X, Y)) = \sum_{j=1}^{n} p_j^2 |z_j|^{2(p_j-1)} |X_j|^2 - q^2 |w|^{2(q-1)} |Y|^2,$$
$$(z, w) \in K_{p,q}, \ (X, Y) \in \mathbb{C}^n \times \mathbb{C},$$

whereas the complex tangent space at $(z, w) \in K_{p,q}$ is given by

$$T_{\mathbb{C}}(z, w) = \left\{ (X, Y) \in \mathbb{C}^n \times \mathbb{C} : Y = \frac{1}{q\overline{w}|w|^{2(q-1)}} \sum_{j=1}^{n} p_j \overline{z}_j |z_j|^{2(p_j-1)} X_j \right\}$$

(recall that $w \neq 0$).

Fix $(z, w) \in K_{p,q}$ such that at least two coordinates of z are nonzero. To see that the Levi form of r restricted to the complex tangent space is not degenerate at (z, w), it suffices to observe that for any $(X, Y) \in T_{\mathbb{C}}(z, w)$

$$\mathcal{L}r((z, w); (X, Y)) = \frac{1}{|w|^{2q}} \sum_{1 \leq j < k \leq n} |z_j|^{2(p_j-1)} |z_k|^{2(p_k-1)} \left| p_j z_k X_j - p_k z_j X_k \right|^2.$$

□

Lemma 14 *Let $D \subset \mathbb{C}^{n+1}$ and $V \subset \mathbb{C}^n$ be bounded domains, $a \in V$, and let $\Phi : V \longrightarrow \partial D$ be a holomorphic mapping such that* rank $\Phi'(a) = n$. *Assume that D has a local defining function r of class $\mathcal{C}^2$ in a neighborhood of $\Phi(a)$. Then ∂D is Levi flat at $\Phi(a)$.*

Proof of Lemma 14 Equality $r(\Phi(z)) = 0$, $z = (z_1, \ldots, z_n) \in V$, implies

$$\sum_{j=1}^{n+1} \frac{\partial r}{\partial z_j}(\Phi(z)) \frac{\partial \Phi_j}{\partial z_m}(z) = 0, \quad z \in V, \ m = 1, \ldots, n, \tag{5}$$

i.e.,

$$X_m(z) := \left(\frac{\partial \Phi_1}{\partial z_m}(z), \ldots, \frac{\partial \Phi_{n+1}}{\partial z_m}(z) \right) \in T_{\mathbb{C}}(\Phi(z)), \quad z \in V, \ m = 1, \ldots, n.$$

Differentiating (5) with respect to $\overline{z}_m$ we get

$$\sum_{j,k=1}^{n+1} \frac{\partial^2 r}{\partial z_j \partial \overline{z}_k}(\Phi(z)) \frac{\partial \Phi_j}{\partial z_m}(z) \overline{\frac{\partial \Phi_j}{\partial z_m}}(z) = 0, \quad z \in V, \ m = 1, \ldots, n.$$

Last equality for $z = a$ gives

$$\mathcal{L}r(\Phi(a); X_m(a)) = 0, \quad m = 1, \ldots, n. \tag{6}$$

On the other hand, rank $\Phi'(a) = n$ implies that the vectors $X_m(a)$, $m = 1, \ldots, n$, form the basis of the complex tangent space $T_{\mathbb{C}}(\Phi(a))$. Consequently, (6) implies that $\mathcal{L}r(\Phi(a); X) = 0$ for any $X \in T_{\mathbb{C}}(\Phi(a))$, i.e., ∂D is Levi flat at $\Phi(a)$. □

Proof of Lemma 11 In view of Lemmas 2.1 and 2.3 from [2] it suffices to consider the case $n \geq 2$ and $m = 1$.

First we show that $F(L_{p,q}) \subset L_{\tilde{p},\tilde{q}} \cup M_{\tilde{p},\tilde{q}}$. Suppose the contrary. Then $F(L_{p,q}) \cap K_{\tilde{p},\tilde{q}} \neq \varnothing$ or $0 \in F(L_{p,q})$.

First assume $F(L_{p,q}) \cap K_{\tilde{p},\tilde{q}} \neq \varnothing$. Since $L_{p,q} \setminus Z(J_F)$ is a dense open set of $L_{p,q}$ (here $Z(J_F)$ denotes the zero set of the Jacobian J_F of a mapping F), the continuity of F implies that there is a point $(z_0, w_0) \in L_{p,q} \setminus Z(J_F)$ such that $F(z_0, w_0) \in K_{\tilde{p},\tilde{q}}$. Without loss of generality, we may assume that at least two coordinates of $G(z_0, w_0)$ are nonzero, where $F(z_0, w_0) = (G(z_0, w_0), H(z_0, w_0)) \in \mathbb{C}^n \times \mathbb{C}$. Consequently, there is an open neighborhood $U \subset \mathbb{C}^n \times \mathbb{C}$ of (z_0, w_0) such that $F|_U : U \longrightarrow F(U)$ is biholomorphic and $F(U \cap L_{p,q}) = F(U) \cap K_{\tilde{p},\tilde{q}}$. Take a neighborhood $V \subset \mathbb{C}^n$ of z_0 such that $(z, w_0) \in U \cap L_{p,q}$ for $z \in V$. Then

$$V \ni z \overset{\Phi}{\longmapsto} F(z, w_0) \in F(U) \cap K_{\tilde{p},\tilde{q}}$$

is a holomorphic mapping with rank $\Phi'(z_0) = n$. By Lemma 14, $K_{\tilde{p},\tilde{q}}$ is Levi flat at $F(z_0, w_0)$, which contradicts Lemma 13.

The assumption $0 \in F(L_{p,q})$ also leads to a contradiction. Indeed, one may repeat the reasoning from the proof of Lemma 2.1 from [2].

Now we shall prove that $F(K_{p,q}) \subset K_{\tilde{p},\tilde{q}} \cup M_{\tilde{p},\tilde{q}}$. Suppose the contrary. Then $F(K_{p,q}) \cap L_{\tilde{p},\tilde{q}} \neq \varnothing$ or $0 \in F(K_{p,q})$.

Suppose $F(K_{p,q}) \cap L_{\tilde{p},\tilde{q}} \neq \varnothing$. Since $K_{p,q} \setminus Z(J_F)$ is a dense open set of $K_{p,q}$, the continuity of F implies that there is a point $(z_0, w_0) \in K_{p,q} \setminus Z(J_F)$ such that $F(z_0, w_0) \in L_{\tilde{p},\tilde{q}}$. Without loss of generality we may assume that at least two coordinates of z_0 are nonzero. Consequently, there is an open neighborhood $U \subset \mathbb{C}^n \times \mathbb{C}$ of (z_0, w_0) such that $F|_U : U \longrightarrow F(U)$ is biholomorphic and $F(U \cap K_{p,q}) = F(U) \cap L_{\tilde{p},\tilde{q}}$. It remains to apply the previous reasoning to the inverse mapping $(F|_U)^{-1} : F(U) \longrightarrow U$.

Finally, the assumption $0 \in F(K_{p,q})$ also leads to a contradiction. Indeed, one may repeat the reasoning from the proof of Lemma 2.3 from [2]. □

5 Proofs of the Theorems 1, 3, and 5

In the proof of Theorem 1 we shall use a part of the main result from [8], where complete characterization of non-elementary proper holomorphic mappings between bounded Reinhardt domains in $\mathbb{C}^2$ is given (cf. [10] for the unbounded case).

Proof of Theorem 1 Observe that (a) and (c) follow immediately from (b).

If $F = (G, H)$ is of the form given in (b), then it is holomorphic and

$$|G(z,w)|^{\tilde{p}}|H(z,w)|^{-\tilde{q}} = \begin{cases} \left(|z||w|^{-q/p}\right)^{k\tilde{p}}, & \text{if } q/p \notin \mathbb{Q} \\ \left(|z||w|^{-q/p}\right)^{k'\tilde{p}} \left|B(z^{p'}w^{-q'})\right|^{\tilde{p}}, & \text{if } q/p \in \mathbb{Q}, \end{cases}$$

i.e., F is proper.

On the other hand, let $F : \mathbb{F}_{p,q} \longrightarrow \mathbb{F}_{\tilde{p},\tilde{q}}$ be an arbitrary mapping which is proper and holomorphic.

Assume first that F is elementary and algebraic, i.e., it is of the form

$$F(z,w) = \left(\alpha z^a w^b, \beta z^c w^d\right),$$

where $a, b, c, d \in \mathbb{Z}$ are such that $ad - bc \neq 0$ and $\alpha, \beta \in \mathbb{C}$ are some constants. Since F is surjective, we infer that $c = 0$, $d \in \mathbb{N}$, and $\xi := \beta \in \mathbb{T}$. Moreover,

$$|\alpha|^{\tilde{p}}|z|^{a\tilde{p}}|w|^{b\tilde{p}-d\tilde{q}} < 1, \tag{7}$$

whence $a \in \mathbb{N}$, $b\tilde{p} - d\tilde{q} \geq 0$, and $\zeta := \alpha \in \mathbb{T}$. Let $k := a$, $l := d$. One may rewrite (7) as

$$\left(|z|^p|w|^{-q}\right)^{k\tilde{p}/p} |w|^{b\tilde{p}-l\tilde{q}+kq\tilde{p}/p} < 1.$$

Taking a sequence $(z_\nu, 1/2)_{\nu\in\mathbb{N}} \subset \mathbb{F}_{p,q}$ with $|z_\nu|^p 2^q \to 1$ as $\nu \to \infty$, we infer that $b\tilde{p} - l\tilde{q} + kq\tilde{p}/p = 0$, i.e.,

$$b = \frac{l\tilde{q}}{\tilde{p}} - \frac{kq}{p}.$$

Consequently, F is as in Theorem 1 (b).

Assume now that F is non-elementary. Then it follows from Theorem 0.1 in [8] that F is of the form

$$F(z,w) = \left(\alpha z^a w^b \tilde{B}\left(z^{p'}w^{-q'}\right), \beta w^l\right),$$

where $a, b \in \mathbb{Z}$, $a \geq 0$, $p', q', l \in \mathbb{N}$, p', q' are relatively prime,

$$\frac{q'}{p'} = \frac{q}{p}, \quad \frac{\tilde{q}}{\tilde{p}} = \frac{aq' + bp'}{lp'}, \tag{8}$$

$\alpha, \beta \in \mathbb{C}$ are some constants, and $\tilde{B}$ is a non-constant finite Blaschke product non-vanishing at the origin.

From the surjectivity of F, we immediately infer that $\zeta := \alpha \in \mathbb{T}$ and $\xi := \beta \in \mathbb{T}$. If we put $k' := a$, then (8) implies

$$b = \frac{l\tilde{q}}{\tilde{p}} - \frac{k'q}{p},$$

which ends the proof. $\square$

Proof of Theorem 3 Firstly, if p, q, $\tilde{p}$, and $\tilde{q}$ satisfy the condition in (a) then the mapping

$$\mathbb{F}_{p,q} \ni (z_1, \dots, z_n, w) \longmapsto \left(z_{\sigma(1)}^{p_{\sigma(1)}/\tilde{p}_1} w^{(r\tilde{q}-q)/\tilde{p}_1}, \dots, z_{\sigma(n)}^{p_{\sigma(n)}/\tilde{p}_n} w^{(r\tilde{q}-q)/\tilde{p}_n}, w^r \right) \in \mathbb{F}_{\tilde{p},\tilde{q}}$$

is proper and holomorphic.

Secondly, if the mapping F is defined by the formulas given in (b), then, using Theorem 9 (b), it is easy to see that $F : \mathbb{F}_{p,q} \longrightarrow \mathbb{F}_{\tilde{p},\tilde{q}}$ is proper and holomorphic.

Finally, (c) is a direct consequence of (b) and Theorem 9 (c).

Thus it remains to prove that if $F : \mathbb{F}_{p,q} \longrightarrow \mathbb{F}_{\tilde{p},\tilde{q}}$ is proper and holomorphic, then p, q, $\tilde{p}$, and $\tilde{q}$ satisfy conditions in (a) and F is given by formulas stated in (b).

Let

$$F = (G, H) = (G_1, \dots, G_n, H) : \mathbb{F}_{p,q} \longrightarrow \mathbb{F}_{\tilde{p},\tilde{q}}$$

be a proper holomorphic mapping. Since $F(L_{p,q}) \subset L_{\tilde{p},\tilde{q}} \cup M_{\tilde{p},\tilde{q}}$ (cf. Lemma 11), it follows from the proof of Lemma 2.2 in [2] that H does not depend on the variable z. Hence $h := H(0, \cdot)$ is a proper and holomorphic self-mapping of $\mathbb{D}_*$. Consequently, by the Hartogs theorem, it extends to a proper holomorphic mapping $h : \mathbb{D} \longrightarrow \mathbb{D}$, i.e., h is a finite Blaschke product. On the other hand, if $h(a) = 0$, we immediately get

$$G(z, a) = 0, \quad \sum_{j=1}^{n} |z_j|^{2p_j} < |a|^{2q},$$

which clearly gives a contradiction, unless $a = 0$. Hence

$$H(z, w) = \xi w^r \tag{9}$$

for some $\xi \in \mathbb{T}$ and $r \in \mathbb{N}$.

For w, $0 < |w| < 1$, let

$$\mathbb{E}_{p,q}(w) := \left\{ (z_1, \dots, z_n) \in \mathbb{C}^n : \sum_{j=1}^{n} |z_j|^{2p_j} < |w|^{2q} \right\}.$$

Since $F(K_{p,q}) \subset K_{\tilde{p},\tilde{q}} \cup M_{\tilde{p},\tilde{q}}$ (cf. Lemma 11), it follows from (9) that $G(\cdot, w) : \mathbb{E}_{p,q}(w) \longrightarrow \mathbb{E}_{\tilde{p},r\tilde{q}}(w)$ is proper and holomorphic. Hence, if we put

$$g_j(z_1, \dots, z_n) := w^{-r\tilde{q}/\tilde{p}_j} G_j \left(z_1 w^{q/p_1}, \dots, z_n w^{q/p_n}, w \right), \quad j = 1, \dots, n,$$

we conclude that $g = (g_1, \dots, g_n) : \mathbb{E}_p \longrightarrow \mathbb{E}_{\tilde{p}}$ is proper and holomorphic. By Theorem 9 (a), there is $\sigma \in \Sigma_n$ such that $p_\sigma / \tilde{p} \in \mathbb{N}^n$. Moreover, it follows from the proof of Theorem 2 in [1] that g does not depend on w. Consequently, we obtain

$$G_j(z_1, \dots, z_n, w) = w^{r\tilde{q}/\tilde{p}_j} g_j \left(z_1 w^{-q/p_1}, \dots, z_n w^{-q/p_n} \right), \quad j = 1, \dots, n.$$

To complete the proof, it remains to make use of the explicit form of the mapping g [cf. Theorem 9 (b)]. □

Proof of Theorem 5 We write $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ and $w = (w_1, \dots, w_m) \in \mathbb{C}^m$. Without loss of generality, we may assume that there is $0 \le \nu \le n$ with $\tilde{p} \in \{1\}^\nu \times (\mathbb{R}_{>0} \setminus \{1\})^{n-\nu}$ and $0 \le \mu \le m$ with $\tilde{q} \in \{1\}^\mu \times (\mathbb{R}_{>0} \setminus \{1\})^{m-\mu}$. Let

$$F = (G, H) : \mathbb{F}_{p,q} \longrightarrow \mathbb{F}_{\tilde{p},\tilde{q}} \subset \mathbb{C}^n \times \mathbb{C}^m$$

be a proper holomorphic mapping. It follows from Lemma 11 that $F(L_{p,q}) \subset L_{\tilde{p},\tilde{q}} \cup M_{\tilde{p},\tilde{q}}$ and hence, using Lemma 2.2 from [2] (note that the proof remains valid for $n = 1$), we infer that H is independent of the variable z. Hence, the mapping $h := H(0, \cdot) : (\mathbb{E}_q)_* \longrightarrow (\mathbb{E}_{\tilde{q}})_*$ is proper and holomorphic. Consequently, by the Hartogs theorem, it extends to a proper and holomorphic mapping $h : \mathbb{E}_q \longrightarrow \mathbb{E}_{\tilde{q}}$, i.e., [cf. Theorem 9 (b)]

$$h = \Psi_{q_\tau/(\tilde{q}t)} \circ \psi \circ \Psi_t \circ \tau$$

for some $\tau \in \Sigma_m$ with $q_\tau/\tilde{q} \in \mathbb{N}^m$, $t \in \mathbb{N}^m$ with $q_\tau/(\tilde{q}t) \in \mathbb{N}^m$, and $\psi \in \mathrm{Aut}(\mathbb{E}_{q_\tau/t})$ with $\psi(0) = 0$. Indeed, if $a = (a_1, \dots, a_m)$ is a zero of h, we immediately get

$$G(z, a) = 0, \quad \sum_{j=1}^{n} |z_j|^{2p_j} < \sum_{j=1}^{m} |a_j|^{2q_j},$$

which is clearly a contradiction, unless $a = 0$. Consequently, $h(0) = 0$.

Without loss of generality, we may assume that there is $\mu \le l' \le m$ with $1/\tilde{q}_j \notin \mathbb{N}$ if and only if $j = l' + 1, \dots, m$. It follows from the proof of Theorem 9 (b) that there is $\mu \le l \le l'$ such that

$$\frac{q_{\tau(j)}}{t_j} = \begin{cases} 1, & \text{if } j = 1, \dots, l \\ \tilde{q}_j, & \text{if } j = l + 1, \dots, m, \end{cases}$$

whence

$$\psi(w) = (U(w_1, \dots, w_l), \xi_{l+1} w_{l+\omega(1)}, \dots, \xi_m w_{l+\omega(m-l)}),$$

where $U = (U_1, \dots, U_l) \in \mathbb{U}(l)$, $\xi_j \in \mathbb{T}$, $j > l$, and $\omega \in \Sigma_{m-l}(\tilde{q}_{l+1}, \dots, \tilde{q}_m)$. Finally,

$$h(w) = \left(U_1^{1/\tilde{q}_1}\left(w_{\tau(1)}^{q_{\tau(1)}}, \dots, w_{\tau(l)}^{q_{\tau(l)}}\right), \dots, U_l^{1/\tilde{q}_l}\left(w_{\tau(1)}^{q_{\tau(1)}}, \dots, w_{\tau(l)}^{q_{\tau(l)}}\right),\right.$$
$$\left.\xi_{l+1} w_{\tau(l+1)}^{q_{\tau(l+1)}/\tilde{q}_{l+1}}, \dots, \xi_m w_{\tau(m)}^{q_{\tau(m)}/\tilde{q}_m}\right).$$

In particular, if we write $h = (h_1, \dots, h_m)$, then

$$\sum_{j=1}^{m} |h_j(w)|^{2\tilde{q}_j} = \sum_{j=1}^{m} |w_j|^{2q_j}, \quad w = (w_1, \dots, w_m) \in \mathbb{E}_q. \tag{10}$$

For $w \in \mathbb{C}^m$, $0 < \rho_w := \sum_{j=1}^{m} |w_j|^{2q_j} < 1$ let

$$\mathbb{E}_{p,q}(w) := \left\{ z \in \mathbb{C}^n : \sum_{j=1}^{n} |z_j|^{2p_j} < \sum_{j=1}^{m} |w_j|^{2q_j} \right\}.$$

Since $F(K_{p,q}) \subset K_{\tilde{p},\tilde{q}} \cup M_{\tilde{p},\tilde{q}}$ (cf. Lemma 11), it follows from (10) that $g := G(\cdot, w) : \mathbb{E}_{p,q}(w) \longrightarrow \mathbb{E}_{\tilde{p},q}(w)$ is proper and holomorphic. Note that g may depend, a priori, on w.

We consider two cases, $n = 1$ and $n \ge 2$, separately.

(i) Case $n = 1$. Here $\mathbb{E}_{p,q}(w) = \rho_w^{1/(2p)}\mathbb{D}$. Consequently,

$$g(z) = \rho_w^{1/(2\tilde{p})} B\left(z\rho_w^{-1/(2p)}\right), \quad z \in \rho_w^{1/(2p)}\mathbb{D}, \tag{11}$$

where B is a finite Blaschke product. Let

$$\mathbb{F}^0_{p,q} := \mathbb{F}_{p,q} \cap \left(\mathbb{C} \times \{0\}^{\tau(1)-1} \times \mathbb{C} \times \{0\}^{m-\tau(1)}\right),$$
$$\mathbb{F}^0_{\tilde{p},q_\tau/t} := \mathbb{F}_{\tilde{p},q_\tau/t} \cap \left(\mathbb{C}^2 \times \{0\}^{m-1}\right).$$

Let $\Phi \in \operatorname{Aut}(\mathbb{F}_{\tilde{p},q_\tau/t})$ be defined by

$$\Phi(z,w) := \left(z, U^{-1}(w_1,\ldots,w_l), w_{l+1},\ldots,w_m\right)$$

and let

$$\hat{\xi}_1 := \begin{cases} \xi_1, & \text{if } l = 0 \\ 1, & \text{if } l > 0 \end{cases}, \qquad \hat{q}_1 := \begin{cases} \tilde{q}_1, & \text{if } l = 0 \\ 1, & \text{if } l > 0. \end{cases}$$

Then $\Phi \circ (G, \psi \circ \Psi_t \circ \tau) : \mathbb{F}^0_{p,q} \longrightarrow \mathbb{F}^0_{\tilde{p},q_\tau/t}$ is proper and holomorphic with

$$(\Phi \circ (G, \psi \circ \Psi_t \circ \tau))(z,w) = \left(G(z,w), \hat{\xi}_1 w_{\tau(1)}^{q_{\tau(1)}/\hat{q}_1}, 0, \ldots, 0\right), \quad (z,w) \in \mathbb{F}^0_{p,q}. \tag{12}$$

It follows from Theorem 1 that

$$(\Phi \circ (G, \psi \circ \Psi_t \circ \tau))(z,w) = \left(\hat{G}(z,w), \eta w^r_{\tau(1)}, 0, \ldots, 0\right), \quad (z,w) \in \mathbb{F}^0_{p,q}, \tag{13}$$

where

$$\hat{G}(z,w) := \begin{cases} \zeta z^k w_{\tau(1)}^{r\hat{q}_1/\tilde{p} - kq_{\tau(1)}/p}, & \text{if } q_{\tau(1)}/p \notin \mathbb{Q} \\ \zeta z^{k'} w_{\tau(1)}^{r\hat{q}_1/\tilde{p} - k'q_{\tau(1)}/p} \hat{B}\left(z^{p'} w_{\tau(1)}^{-q'_{\tau(1)}}\right), & \text{if } q_{\tau(1)}/p \in \mathbb{Q}, \end{cases}$$

$\zeta, \eta \in \mathbb{T}$, $k, r, p', q'_{\tau(1)} \in \mathbb{N}$, $k' \in \mathbb{N} \cup \{0\}$ are such that $p', q'_{\tau(1)}$ are relatively prime, $q_{\tau(1)}/p = q'_{\tau(1)}/p'$, $r\hat{q}_1/\tilde{p} - kq_{\tau(1)}/p \in \mathbb{Z}$, and $\hat{B}$ is a finite Blaschke product non-vanishing at 0 (if $\hat{B} \equiv 1$, then $k' > 0$). Hence

$$\begin{aligned}&(\Phi \circ (G, \psi \circ \Psi_t \circ \tau))(z,w) \\ &\quad = \left(\hat{G}(z,w) + \alpha(z,w), w_{\tau(1)}^{q_{\tau(1)}}, \ldots, w_{\tau(l)}^{q_{\tau(l)}}, \xi_{l+1} w_{\tau(l+1)}^{q_{\tau(l+1)}/\tilde{q}_{l+1}}, \ldots, \xi_m w_{\tau(m)}^{q_{\tau(m)}/\tilde{q}_m}\right),\end{aligned} \tag{14}$$

for $(z,w) \in \mathbb{F}_{p,q}$, $w_{\tau(1)} \neq 0$, where α is holomorphic on $\mathbb{F}_{p,q}$ with $\alpha|_{\mathbb{F}^0_{p,q}} = 0$. Comparing (12) and (13) we conclude that

$$\eta = \hat{\xi}_1, \quad r = q_{\tau(1)}/\hat{q}_1.$$

Since the mapping on the left side of (14) is holomorphic on $\mathbb{F}_{p,q}$, the function

$$\hat{G}(z,w) = \begin{cases} \zeta z^k w_{\tau(1)}^{q_{\tau(1)}(1/\tilde{p} - k/p)}, & \text{if } q_{\tau(1)}/p \notin \mathbb{Q} \\ \zeta z^{k'} w_{\tau(1)}^{q_{\tau(1)}(1/\tilde{p} - k'/p)} \hat{B}\left(z^{p'} w_{\tau(1)}^{-q'_{\tau(1)}}\right), & \text{if } q_{\tau(1)}/p \in \mathbb{Q} \end{cases} \tag{15}$$

with $q_{\tau(1)}(1/\tilde{p} - k/p) \in \mathbb{Z}$ and $q_{\tau(1)}(1/\tilde{p} - k'/p) \in \mathbb{Z}$ has to be holomorphic on $\mathbb{F}_{p,q}$, as well. Since $m \geq 2$, it may happen $w_{\tau(1)} = 0$. Consequently, $q_{\tau(1)}(1/\tilde{p} - k/p) \in \mathbb{N} \cup \{0\}$ in the first case of (15), whereas $\hat{B}(t) = t^{k''}$ for some $k'' \in \mathbb{N}$ with $q_{\tau(1)}(1/\tilde{p} - k'/p) - k''q'_{\tau(1)} \in \mathbb{N} \cup \{0\}$ in the second case. Thus

$$\hat{G}(z,w) = \zeta z^k w_{\tau(1)}^{q_{\tau(1)}(1/\tilde{p} - k/p)},$$

where $k \in \mathbb{N}$, $q_{\tau(1)}(1/\tilde{p} - k/p) \in \mathbb{N} \cup \{0\}$ [in the second case of (15) it suffices to take $k := k' + p'k''$].

Observe that $\hat{G} + \alpha = G$. Fix $w \in \{0\}^{\tau(1)-1} \times \mathbb{C} \times \{0\}^{m-\tau(1)}$ with $0 < \rho_w < 1$. Then $\rho_w = |w_{\tau(1)}|^{2q_{\tau(1)}}$ and $\hat{G}(\cdot, w) = g$ on $\rho_w^{1/(2p)}\mathbb{D}$, i.e.,

$$\zeta z^k w_{\tau(1)}^{q_{\tau(1)}(1/\tilde{p}-k/p)} = |w_{\tau(1)}|^{q_{\tau(1)}/\tilde{p}} B\left(z|w_{\tau(1)}|^{-q_{\tau(1)}/p}\right), \quad z \in |w_{\tau(1)}|^{q_{\tau(1)}/p}\mathbb{D}.$$

Thus $B(t) = \zeta t^k$ and $q_{\tau(1)}(1/\tilde{p} - k/p) = 0$, i.e., $k = p/\tilde{p}$. Hence part (a) in the case $n = 1$ is proved. To finish part (b) in this case, note that $g(z) = \zeta z^{p/\tilde{p}}$. Consequently, g does not depend on w and

$$G(z, w) = \zeta z^{p/\tilde{p}}, \quad (z, w) \in \mathbb{F}_{p,q}.$$

(ii) Case $n \geq 2$. Let

$$f_j(z) := \rho_w^{-1/(2\tilde{p}_j)} g_j\left(z_1 \rho_w^{1/(2p_1)}, \ldots, z_n \rho_w^{1/(2p_n)}\right), \quad j = 1, \ldots, n. \tag{16}$$

Then $f := (f_1, \ldots, f_n) : \mathbb{E}_p \longrightarrow \mathbb{E}_{\tilde{p}}$ is proper and holomorphic, i.e.,

$$f = \Psi_{p_\sigma/(\tilde{p}s)} \circ \varphi \circ \Psi_s \circ \sigma \tag{17}$$

for some $\sigma \in \Sigma_n$ with $p_\sigma/\tilde{p} \in \mathbb{N}^n$, $s \in \mathbb{N}^n$ with $p_\sigma/(\tilde{p}s) \in \mathbb{N}^n$, and $\varphi \in \mathrm{Aut}(\mathbb{E}_{p_\sigma/s})$. In particular, part (a) in the case $n \geq 2$ is proved. Without loss of generality, we may assume that there is $\nu \leq k' \leq n$ such that $1/\tilde{p}_j \notin \mathbb{N}$ if and only if $j = k'+1, \ldots, n$. It follows from the proof of Theorem 9 (b) that there is $\nu \leq k \leq k'$ such that

$$\frac{p_{\sigma(j)}}{s_j} = \begin{cases} 1, & \text{if } j = 1, \ldots, k \\ \tilde{p}_j, & \text{if } j = k+1, \ldots, n, \end{cases}$$

whence

$$\varphi(z) = \left(T(z'), \zeta_{k+1} z_{k+\omega(1)} \left(\frac{\sqrt{1 - \|a'\|^2}}{1 - \langle z', a' \rangle}\right)^{1/\tilde{p}_{k+\omega(1)}}, \ldots, \zeta_n z_{k+\omega(n-k)} \left(\frac{\sqrt{1 - \|a'\|^2}}{1 - \langle z', a' \rangle}\right)^{1/\tilde{p}_{k+\omega(n-k)}}\right),$$

where $T = (T_1, \ldots, T_k) \in \mathrm{Aut}(\mathbb{B}_k)$, $z' := (z_1, \ldots, z_k)$, $a' := T^{-1}(0)$, $\zeta_j \in \mathbb{T}$, $j > k$, and $\omega \in \Sigma_{n-k}(\tilde{p}_{k+1}, \ldots, \tilde{p}_n)$. Let

$$\mathbb{F}^0_{p,q} := \mathbb{F}_{p,q} \cap \left(\mathbb{C}^n \times \{0\}^{\tau(1)-1} \times \mathbb{C} \times \{0\}^{m-\tau(1)}\right),$$
$$\mathbb{F}^0_{\tilde{p},q_\tau/t} := \mathbb{F}_{\tilde{p},q_\tau/t} \cap \left(\mathbb{C}^{n+1} \times \{0\}^{m-1}\right).$$

Let $\Phi \in \mathrm{Aut}(\mathbb{F}_{\tilde{p},q_\tau/t})$ be defined by

$$\Phi(z, w) := \left(z, U^{-1}(w_1, \ldots, w_l), w_{l+1}, \ldots, w_m\right)$$

and let

$$\hat{\xi}_1 := \begin{cases} \xi_1, & \text{if } l = 0 \\ 1, & \text{if } l > 0 \end{cases}, \qquad \hat{q}_1 := \begin{cases} \tilde{q}_1, & \text{if } l = 0 \\ 1, & \text{if } l > 0. \end{cases}$$

Then $\Phi \circ (G, \psi \circ \Psi_t \circ \tau) : \mathbb{F}^0_{p,q} \longrightarrow \mathbb{F}^0_{\tilde{p},q_\tau/t}$ is proper and holomorphic with

$$(\Phi \circ (G, \psi \circ \Psi_t \circ \tau))(z, w) = \left(G(z, w), \hat{\xi}_1 w^{q_{\tau(1)}/\hat{q}_1}_{\tau(1)}, 0, \dots, 0\right), \quad (z, w) \in \mathbb{F}^0_{p,q}. \tag{18}$$

It follows from Theorem 3 (b) that

$$(\Phi \circ (G, \psi \circ \Psi_t \circ \tau))(z, w) = \left(\hat{G}(z, w), \eta w^r_{\tau(1)}, 0, \dots, 0\right), \quad (z, w) \in \mathbb{F}^0_{p,q}, \tag{19}$$

where $\hat{G} = (\hat{G}_1, \dots, \hat{G}_n)$,

$$\hat{G}_j(z, w) := w^{r\hat{q}_1/\tilde{p}_j}_{\tau(1)} \hat{f}_j \left(z_1 w^{-q_{\tau(1)}/p_1}_{\tau(1)}, \dots, z_n w^{-q_{\tau(1)}/p_n}_{\tau(1)}\right), \quad j = 1, \dots, n,$$

$\eta \in \mathbb{T}$, $r \in \mathbb{N}$, and $\hat{f} := (\hat{f}_1, \dots, \hat{f}_n) : \mathbb{E}_p \longrightarrow \mathbb{E}_{\tilde{p}}$ is proper and holomorphic, i.e.,

$$\hat{f} = \Psi_{p\hat{\sigma}/(\tilde{p}\hat{s})} \circ \hat{\varphi} \circ \Psi_{\hat{s}} \circ \hat{\sigma} \tag{20}$$

for some $\hat{\sigma} \in \Sigma_n$ with $p_{\hat{\sigma}}/\tilde{p} \in \mathbb{N}^n$, $\hat{s} \in \mathbb{N}^n$ with $p_{\hat{\sigma}}/(\tilde{p}\hat{s}) \in \mathbb{N}^n$, and $\hat{\varphi} \in \mathrm{Aut}(\mathbb{E}_{p_{\hat{\sigma}}}/\hat{s})$. Again, it follows from the proof of Theorem 9 (b) that

$$\frac{p_{\hat{\sigma}(j)}}{\hat{s}_j} = \begin{cases} 1, & \text{if } j = 1, \dots, k \\ \tilde{p}_j, & \text{if } j = k+1, \dots, n, \end{cases}$$

whence

$$\hat{\varphi}(z) = \left(\hat{T}(z'), \hat{\zeta}_{k+1} z_{k+\hat{\omega}(1)} \left(\frac{\sqrt{1 - \|\hat{a}'\|^2}}{1 - \langle z', \hat{a}'\rangle}\right)^{1/\tilde{p}_{k+\hat{\omega}(1)}}, \right.$$
$$\left. \dots, \hat{\zeta}_n z_{k+\hat{\omega}(n-k)} \left(\frac{\sqrt{1 - \|\hat{a}'\|^2}}{1 - \langle z', \hat{a}'\rangle}\right)^{1/\tilde{p}_{k+\hat{\omega}(n-k)}}\right),$$

where $\hat{T} = (\hat{T}_1, \dots, \hat{T}_k) \in \mathrm{Aut}(\mathbb{B}_k)$, $z' := (z_1, \dots, z_k)$, $\hat{a}' := \hat{T}^{-1}(0)$, $\hat{\zeta}_j \in \mathbb{T}$, $j > k$, and $\hat{\omega} \in \Sigma_{n-k}(\tilde{p}_{k+1}, \dots, \tilde{p}_n)$. From (19) we infer that

$$\begin{aligned}&(\Phi \circ (G, \psi \circ \Psi_t \circ \tau))(z, w) \\ &= \left(\hat{G}(z, w) + \alpha(z, w), w^{q_{\tau(1)}}_{\tau(1)}, \dots, w^{q_{\tau(l)}}_{\tau(l)}, \xi_{l+1} w^{q_{\tau(l+1)}/\tilde{q}_{l+1}}_{\tau(l+1)}, \dots, \xi_m w^{q_{\tau(m)}/\tilde{q}_m}_{\tau(m)}\right),\end{aligned} \tag{21}$$

for $(z, w) \in \mathbb{F}_{p,q}$ with $w_{\tau(1)} \neq 0$, where α is holomorphic on $\mathbb{F}_{p,q}$ with $\alpha|_{\mathbb{F}^0_{p,q}} = 0$. Comparing (18) and (19) we conclude that

$$\eta = \hat{\xi}_1, \quad r = q_{\tau(1)}/\hat{q}_1.$$

Since the mapping on the left side of (21) is holomorphic on $\mathbb{F}_{p,q}$, the functions

$$\hat{G}_j(z, w) = \begin{cases} w^{q_{\tau(1)}/\tilde{p}_j}_{\tau(1)} \hat{T}^{1/\tilde{p}_j}_j \left(z^{p_{\hat{\sigma}(1)}}_{\hat{\sigma}(1)} w^{-q_{\tau(1)}}_{\tau(1)}, \dots, z^{p_{\hat{\sigma}(k)}}_{\hat{\sigma}(k)} w^{-q_{\tau(1)}}_{\tau(1)}\right), & \text{if} j \leq k \\ \hat{\zeta}_j z^{p_{\hat{\sigma}(j)}/\tilde{p}_j}_{\hat{\sigma}(j)} \left(\frac{\sqrt{1-\|\hat{a}'\|^2}}{1-\langle z', \hat{a}'\rangle}\right)^{1/\tilde{p}_{\hat{\sigma}(j)}}, & \text{if } j > k, \end{cases}$$

are holomorphic on $\mathbb{F}_{p,q}$, as well. Since $m \geq 2$, it may happen $w_{\tau(1)} = 0$. Consequently, $\hat{T} \in \mathbb{U}(k)$ and

$$\hat{G}_j(z,w) = \begin{cases} \hat{T}_j^{1/\tilde{p}_j}\left(z_{\hat{\sigma}(1)}^{p_{\hat{\sigma}(1)}}, \ldots, z_{\hat{\sigma}(k)}^{p_{\hat{\sigma}(k)}}\right), & \text{if } j \leq k \\ \hat{\zeta}_j z_{\hat{\sigma}(j)}^{p_{\hat{\sigma}(j)}/\tilde{p}_j} & \text{if } j > k. \end{cases}$$

Recall that $\hat{G} + \alpha = G$ and fix $w \in \{0\}^{\tau(1)-1} \times \mathbb{C} \times \{0\}^{m-\tau(1)}$ with $0 < \rho_w < 1$. Then $\rho_w = |w_{\tau(1)}|^{2q_{\tau(1)}}$ and it follows from (16) and (17) that

$$g_j(z) = \begin{cases} |w_{\tau(1)}|^{q_{\tau(1)}/\tilde{p}_j} T_j^{1/\tilde{p}_j}\left(z_{\sigma(1)}^{p_{\sigma(1)}}|w_{\tau(1)}|^{-q_{\tau(1)}}, \ldots, z_{\sigma(k)}^{p_{\sigma(k)}}|w_{\tau(1)}|^{-q_{\tau(1)}}\right), & \text{if } j \leq k \\ \zeta_j z_{\sigma(j)}^{p_{\sigma(j)}/\tilde{p}_j}, & \text{if } j > k, \end{cases}$$

From the equality $\hat{G}(\cdot, w) = g$ on $\mathbb{E}_{p,q}(w)$ one has $\zeta_j = \hat{\zeta}_j$, $j > k$, and, losing no generality, we conclude that $\sigma = \hat{\sigma}$, $s = \hat{s}$, and $T = \hat{T}$. Consequently, g does not depend on w and $g^{-1}(0) = 0$.

Part (c) follows directly from (b). $\square$

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